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Relativistic kinetic theory of a system of particles with variable rest mass

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Abstract. In this paper we have endeavoured to study, by the formulation of a relativistic kinetic theory, a system in which the particles have variable rest mass. Macroscopic quantities and their evolution equations have been directly linked to the action responsible for the variation of the rest mass of the particles. The dissipative character of this 'force' gives a slight modification to the Liouville theorem and the Boltzmann equation. A detailed study of reversible processes has shown that the macroscopic quantities keep the same form as for the usual perfect fluid (no bulk viscosity appears) though they are not conserved and can give rise to shearing and vortices. This pattern could serve as a model for the study of certain dissipative processes where mass is lost.

1. Introduction

Until the present time, work done on relativistic kinetic theory has been used to study systems of particles with constant rest mass m_0 (besides collision processes where m_0 may vary). In this paper we propose a new hypothesis, the possibility of variation in the rest mass under the influence of an external field and see what kind of modifications are then made to the usual pattern of relativistic kinetic theory.

Is it possible to justify this new hypothesis, and what kind of applications can we expect it to give? First of all, let us note that the variation of rest mass can only exist if the particle, considered here as a quasi-punctual object, is able to lose mass or internal energy i.e. is capable of internal modifications. Such phenomena may occur if the particles are macroscopic objects as is the case for the kinetic theory approach to cosmology. Fundamental particles seem, at first sight, to be excluded from this pattern. Nevertheless, de Broglie (1964) considers a variation of m_0 in the framework of the hidden thermodynamics of the particle, which will lead to a reinterpretation of quantum mechanics. On the other hand, some authors (Arzeliès 1968, 1971, Costa de Beauregard 1972, Cavalleri and Salgarelli 1969, Landsberg 1970, Møller 1967) have used the variation of the rest mass of the volume element within the framework of relativistic mechanics and thermodynamics of continuous media. Between these two points of view, Fronteau (1973) defines a 'fine' entropy for the particle which varies with m_0 just as a 'fine' temperature.

Our purpose is to consider a variation of the rest mass of the particle and then deduce the macroscopic properties of the system thanks to the formalism of relativistic kinetic theory. The variation of m_0 can be due either to a direct exchange of matter or to an exchange of radiation between the particle and its surroundings (for example, transition between two energy levels). On the whole the system will behave like an open system unless inclusive supplementary conditions are added. This pattern can apply to any dissipative system (thermodynamics, stars with variable mass, steady-state cosmologies) provided that we know the action responsible for the variation of the rest mass of the particles.

We will adopt here the description of relativistic kinetic theory proposed by Synge (1957) and further developed by many authors, among whom are Tauber and Weinberg (1961), Chernikov (1963, 1964a, b), Israël (1963, 1972), Marle (1969), Ehlers (1971) and Stewart (1971).

After briefly returning in § 2 to the equation of motion for a particle with variable rest mass, we will present the pattern of kinetic theory in §§§ 3, 4 and 5 and we will limit this to a study of reversible processes in § 6. We will present a classical study of the particles without taking the quantum effects into account.

Most of the authors previously mentioned have already developed the case of the quantum particle and irreversible processes. Israël in particular studied in detail different sorts of collisions (elastic, inelastic, fusion, fission). He calculated the stresses and the transport coefficients which appear when there are slight departures from equilibrium. We shall see here that, in equilibrium, rest mass variation does not produce viscosity nor supplementary flows and that the energy–momentum tensor will remain that of a perfect gas although shearing and vortices are possible.

2. Equation of motion for a particle with variable rest mass

The manifold V_4 of general relativity is provided with the hyperbolic metric $g_{\alpha\beta}$ of signature $+---$ (Greek subscripts take values 0, 1, 2, 3 and Latin subscripts 1, 2, 3). The general equation of motion for a particle with variable rest mass m_0 and four-velocity u^α ($u^\alpha u_\alpha = 1$) is given by (Henry and Barrabès 1972, Møller 1972):

$$\mathrm{D}p^\alpha/\mathrm{d}s = F^\alpha + f^\alpha, \qquad (2.1)$$

or by the two relationships obtained from (2.1)†:

$$m_0 c(\mathrm{D}u^\alpha/\mathrm{d}s) = F^\alpha + h^\alpha{}_\beta f^\beta$$
$$\mathrm{d}p/\mathrm{d}s = f^\alpha u_\alpha \qquad (2.2)$$

where $p^\alpha = m_0 c u^\alpha$ is the four-momentum, $p = (p_\alpha p^\alpha)^{1/2} = m_0 c$ and where $h^\alpha{}_\beta = \delta^\alpha{}_\beta - u^\alpha u_\beta$ is the projection operator of the subspace orthogonal to the four-velocity. In equations (2.1) and (2.2), F^α represents the sum of all external forces which do not produce a variation of rest mass, and f^α the sum of those which do produce it. We notice in equation (2.2) that F^α is necessarily orthogonal to u^α and that f^α possesses a component along the four-velocity which is different from zero.

One can then consider the following special cases.

2.1. Exchange of particles

If an exchange of particles with rest mass μ takes place, with four-velocity A^α and with a rate of exchange N, then equation (2.4) can be written:

$$\mathrm{D}p^\alpha/\mathrm{d}s = F^\alpha + c\mu N A^\alpha. \qquad (2.3)$$

† D/ds indicates the covariant derivative with respect to s.

A^α is in fact the barycentric velocity of the $N\,\mathrm{d}\tau$ particles which are exchanged during the proper time interval $\mathrm{d}\tau$. For photons with frequency ν, one only has to replace μA^α by $(h\nu/c^2)k^\alpha$ where k^α is an isotropic vector. From equation (2.2) we obtain:

$$\frac{\mathrm{D}p^\alpha}{\mathrm{d}s} = F^\alpha + \frac{p}{A^\beta p_\beta}\frac{\mathrm{d}p}{\mathrm{d}s}A^\alpha. \tag{2.4}$$

2.2. Isotropic exchange

Such an exchange must not produce variation of velocity if the external forces F^α are null. This condition is satisfied if:

$$f^\alpha = c\frac{\mathrm{d}m_0}{\mathrm{d}s}u^\alpha = \frac{\mathrm{d}p}{\mathrm{d}s}\frac{p^\alpha}{p}, \tag{2.5}$$

which gives the equation of motion:

$$\frac{\mathrm{D}p^\alpha}{\mathrm{d}s} = F^\alpha + \frac{\mathrm{d}p}{\mathrm{d}s}\frac{p^\alpha}{p}. \tag{2.6}$$

Remark. One sometimes uses the following variation law of p:

$$\mathrm{d}p/\mathrm{d}s = -\lambda p \tag{2.7}$$

where $\lambda = cte$. In the case of isotropic exchanges we will have:

$$f^\alpha = -\lambda p^\alpha. \tag{2.8}$$

2.3. Case where $f^\alpha = p,^\alpha$

We can greatly simplify the initial equation by using the conformal transformation:

$$\mathrm{d}s^2 \to \widetilde{\mathrm{d}s}^2 = (p/p_0)^2\,\mathrm{d}s^2$$

where $p_0 = cte$. In this metric, equation (1) becomes:

$$[\mathrm{d}(p_0\tilde{u}^\alpha)/\widetilde{\mathrm{d}s}] + \tilde{\Gamma}^\alpha_{\mu\nu}p_0\tilde{u}^\alpha\tilde{u}^\nu = \tilde{F}^\alpha.$$

It is the equation of motion of a particle whose rest mass is $p_0 = cte$ because $\tilde{F}^\alpha\tilde{u}_\alpha = (p/p_0)^2F^\alpha u_\alpha = 0$. Thus when described in terms of the unphysical metrics $\widetilde{\mathrm{d}s}^2$, the particles have constant rest mass and the standard results of kinetic theory hold.

3. Phase space

3.1. Definition

As it is free from the condition $p = cte$, the phase space P_8 is an eight-dimensional fibre bundle with base V_4. The fibre is the set of future-directed, time-like vectors p^α, and the structural group is the orthochronous Lorentz group. We shall note $X^a = (x^\alpha, p^\alpha)$ the coordinates of the generic point of P_8, $(a = 0, 1, \ldots, 7)$:

$$X^a = (X^\alpha = x^\alpha, X^{\alpha+4} = p^\alpha). \tag{3.1}$$

According to § 2 the point X moves in P_8 along a trajectory, the tangent vector L^a which is:

$$L^a = \mathrm{d}X^a/\mathrm{d}s. \tag{3.2}$$

This is only true if the particles do not interact. The components of L^a according to equation (2.1) are:

$$\begin{aligned} L^\alpha &= (p^\alpha/p) \\ L^{\alpha+4} &= F^\alpha + f^\alpha - \Gamma^\alpha_{\beta\gamma}(p^\beta p^\gamma/p)Y^\alpha. \end{aligned} \tag{3.3}$$

The symbols $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of V_4. It is possible to take as volume element on P_8:

$$\Omega = \eta \wedge \pi, \tag{3.4}$$

where η and π are the volume element forms on V_4 and its tangent space respectively:

$$\begin{aligned} \eta &= (-g)^{1/2}\epsilon_{\alpha\beta\mu\nu}\,\mathrm{d}x^{\alpha\beta\mu\nu}/4! = (-g)^{1/2}\,\mathrm{d}x^{0123} \\ \pi &= (-g)^{1/2}\epsilon_{\alpha\beta\mu\nu}\,\mathrm{d}p^{\alpha\beta\mu\nu}/4! = (-g)^{1/2}\,\mathrm{d}p^{0123} \end{aligned} \tag{3.5}$$

This gives:

$$\Omega = |g|\,\mathrm{d}x^{0123} \wedge \mathrm{d}p^{0123}. \tag{3.6}$$

According to the definition of L^a, if $f(x, p)$ is a function defined on P_8, let us note:

$$L(f) = \frac{p^\alpha}{p}\frac{\partial f}{\partial x^\alpha} + Y^\alpha \frac{\partial f}{\partial p^\alpha}. \tag{3.7}$$

3.2. Liouville theorem

We are now going to show that the dissipative character of the equation of motion (2.1) leads to a non-conservation of the volume element of P_8, whereas there was conservation for particles with constant rest mass.

Let us then take the Lie derivative of Ω along the vector field L. Thanks to the well known property of differential forms:

$$\pounds_L\Omega = \mathrm{d}(L\,.\,\Omega) + L\,.\,\mathrm{d}\Omega,$$

where $L\,.\,\Omega$ denotes the interior product of Ω by L and d the exterior differentiation, there remains:

$$\pounds_L\Omega = \mathrm{d}\omega,$$

where we noted $\omega \equiv L\,.\,\Omega$:

$$\omega = -\frac{g}{3!}\left(\frac{p^\alpha}{p}\epsilon_{\alpha\beta\mu\nu}\,\mathrm{d}x^{\beta\mu\nu}\Lambda\,\mathrm{d}p^{0123} + Y^\alpha\epsilon_{\alpha\beta\mu\nu}\,\mathrm{d}x^{0123}\Lambda\,\mathrm{d}p^{\beta\mu\nu}\right).$$

The exterior differentiation of ω gives:

$$\mathrm{d}\omega = \Omega\left(\frac{g_\alpha p^\alpha}{gp} - \frac{p^\alpha p^\mu p^\nu}{2p^3}g_{\mu\nu,\alpha} + \frac{\partial Y^\alpha}{\partial p^\alpha}\right),$$

so the Liouville theorem is written:

$$\pounds_L\Omega = \Omega(\partial(F^\alpha + f^\alpha)/\partial p^\alpha). \tag{3.8}$$

In many cases the derivative of F^α is null. For electromagnetic forces, this is due to the antisymmetry of the electromagnetic field tensor; very often F^α depends neither on p^α nor on p. There remains then:

$$\pounds_L \Omega = \Omega(\partial f^\alpha / \partial p^\alpha). \tag{3.9}$$

The volume element is thus no longer preserved, since generally f^α depends on p^α. By a similar calculation one obtains, since $L \,.\, \omega = 0$:

$$\pounds_L \omega = \omega(\partial(F^\alpha + f^\alpha)/\partial p^\alpha). \tag{3.10}$$

It is possible to calculate the expressions of equation (3.9) in the different special cases already considered. We will find:

(i) Exchange of particles:

$$\pounds_L \Omega = \Omega\left[\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}s} + \frac{pA^\alpha}{p^\beta A_\beta}\frac{\partial}{\partial p^\alpha}\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right) - \frac{p}{(A^\alpha p_\alpha)^2}\frac{\mathrm{d}p}{\mathrm{d}s}\right] \tag{3.11}$$

(ii) Isotropic exchange:

$$\pounds_L \Omega = \Omega\left[\frac{3}{p}\frac{\mathrm{d}p}{\mathrm{d}s} + \frac{p^\alpha}{p}\frac{\partial}{\partial p^\alpha}\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right)\right] \tag{3.12}$$

Remark. When the mass variation is given by equation (2.7), we have for the respective exchanges:

$$\pounds_L \Omega = \lambda\Omega\left(-2 + \frac{p^2}{(A^\alpha p_\alpha)^2}\right), \tag{3.13}$$

$$\pounds_L \Omega = -4\lambda\Omega. \tag{3.14}$$

4. The Boltzmann equation

The history of the medium is shown in the phase space by a set of trajectories whose vertices correspond to collisions between particles (the disappearance of a particle is represented by the end of its trajectory). We shall suppose that the particles only interact by collisions, and we shall limit our study to binary collisions. Since the knowledge of each particle's state cannot be used directly to find the macroscopic properties of the system, a distribution function is introduced on the occupied states of P_8. This will be done in a way similar to the case where the rest mass is constant, but will lead to a slightly different Boltzmann equation (for this section, see Marle 1969, Ehlers 1971, Stewart 1971).

4.1. Distribution function

Given Σ to be an oriented hypersurface of P_8 whose projection on V_4 is a three-dimensional space-like surface, let us take $(A_1, \ldots, A_7)$ as a set of seven linearly independent vectors forming a basis for Σ.

For each hypersurface Σ of this kind, let us suppose that there exists a differential form θ which, when integrated on Σ, gives the numerical flux of the trajectories cutting

Σ (allowing for the orientation of Σ):

$$N(\Sigma) = \int_{\Sigma} \theta.$$

As $\theta(A_1, \ldots, A_7) \neq 0$ and as the vectors $(L, A_1, \ldots, A_7)$ are linearly independent, one has $\theta = f\omega$, and then:

$$N(\Sigma) = \int_{\Sigma} f\omega, \tag{4.1}$$

where $\omega = L \cdot \Omega$ and f is a function of (x^α, p^α). If there are no interactions between particles (no collisions), then θ must be invariant along the trajectories, i.e.:

$$£_L \theta = 0.$$

Taking into account equation (3.10), this gives:

$$L(f) + f\frac{\partial(f^\alpha + F^\alpha)}{\partial p^\alpha} = 0, \tag{4.2}$$

where $L(f)$ is given by equation (3.7).

If we suppress f^α in equation (4.2), we find once more the usual condition $L(f) = 0$ for a system of non-colliding particles with constant rest mass, if $\partial F^\alpha/\partial p^\alpha = 0$. The function f is the distribution function, and is by construction a Lorentz invariant.

If, instead of Σ we take the frontier ∂D of closed domain D of P_8 constituted by two hypersurfaces Σ_1, Σ_2 similar to Σ, and a tube containing the trajectories, one interprets:

$$N(\partial D) = \int_{\partial D} f\omega, \tag{4.3}$$

as the average numerical count of collisions between Σ_1 and Σ_2. Using Stokes' theorem, and taking the relation $\mathrm{d}f \wedge L \cdot \Omega = L(f)\Omega$, equation (4.3) can be written:

$$N(\partial D) = \int_{v} \eta \int_{w} \left(L(f) + f\frac{\partial(f^\alpha + F^\alpha)}{\partial p^\alpha}\right)\pi, \tag{4.4}$$

where the integrals are taken in the domains of space–time v and of four-momentum w included in D. If the numerical count is null for all v (collisions which preserve the numbers of particles), then:

$$\int_{w} \left(L(f) + f\frac{\partial(f^\alpha + F^\alpha)}{\partial p^\alpha}\right)\pi = 0. \tag{4.5}$$

Where $f^\alpha = 0$ and $\partial F^\alpha/\partial p^\alpha = 0$ we have once again the well known relationship for systems of particles with constant rest mass, when the numerical count of collisions is null:

$$\int_{w} L(f)\pi = 0.$$

4.2. Collision hypothesis

Our knowledge of the nature of the collisions will give us the evolution of $f(x^\alpha, p^\alpha)$, represented in this case by the Boltzmann equation:

$$L(f)+f\frac{\partial(f^{\alpha}+F^{\alpha})}{\partial p^{\alpha}}=C[f], \tag{4.6}$$

where $C[f]$, the collision term, is not specified.

In order to simplify, let us take the case of a system with only one sort of particle. If p^{α}, q^{α} (p'^{α}, q'^{α}) are the incident (emergent) four-momenta of two colliding particles, then the conservation law:

$$p^{\alpha}+q^{\alpha}=p'^{\alpha}+q'^{\alpha},$$

indicates that we may equally well be dealing with an elastic as an inelastic collision, when one admits that the rest mass may be variable. The collision is elastic if we also have: $p=p'$ and $q=q'$.

Where $m_0=cte$, it was necessary to introduce at least two sorts of particles in order to be able to use the term 'inelastic collisions'. If we settle for this sort of collision, then the collision term is rewritten (see Israël 1963, 1972 for a detailed study of collision processes):

$$C[f]=\tfrac{1}{2}\int\int\int (f(p'^{\alpha})f(q'^{\alpha})-f(p^{\alpha})f(q^{\alpha}))A_{pq\to p'q'}\pi_q\pi_{p'}\pi_{q'} \tag{4.7}$$

where the integrals are taken on the four-momentum spaces of the particles q, p', q', and where $A_{pq\to p'q'}$ is linked to the probability of the reaction. We take the usual conditions of equation (4.7) to be valid: (i) the gas is not too dense, (ii) the four-momenta (velocity and rest mass) are uncorrelated, (iii) the collisions take place in small space–time volumes so as not to modify the curvature, (iv) microscopic reversibility occurs, i.e.:

$$A_{pq\to p'q'}=A'_{p'q'\to pq}. \tag{4.8}$$

One can check that, since the numerical count is null during these collisions, the expression (4.7) of $C[f]$ gives us equation (4.5). This is a straightforward result taking into account the hypothesis of microscopic reversibility.

If one also admits fusion ($p^{\alpha}+q^{\alpha}=p'^{\alpha}$) and fission ($p^{\alpha}=p'^{\alpha}+q'^{\alpha}$) one must add to the right-hand side of (4.7) (cf Marle 1969, Stewart 1971, Israël 1963, 1972):

$$\int\int (f(p'^{\alpha})-f(p^{\alpha})f(q^{\alpha}))A_{pq\to p'}\pi_{p'}\pi_q+\tfrac{1}{2}\int\int (f(p'^{\alpha})f(q'^{\alpha})-f(p^{\alpha}))A_{p\to p'q'}\pi_{p'}\pi_{q'}.$$

The numerical count of the collisions will no longer be null and equation (4.5) will no longer be valid. The generalization for a multi-component system is straightforward, and will not be made here.

5. Macroscopic quantities and their conservation laws

Having a distribution function on the phase space P_8, one can build macroscopic quantities for the system, and give the corresponding conservation laws.

5.1. *Macroscopic quantities*

Let us build the following quantities:

$$A = \int_w f\pi/p \tag{5.1}$$

$$N^\alpha = \int_w fp^\alpha \pi/p \tag{5.2}$$

$$T^{\alpha\beta} = \int_w fp^\alpha p^\beta \pi/p \tag{5.3}$$

etc . . . (the integrals are taken over the whole four-momentum space w). If the first of these quantities does not seem to have a physical interpretation we can recognize in N^α and $T^{\alpha\beta}$ respectively, the numerical flux vector and the energy–momentum tensor. Indeed, $N(\Sigma)$ appears in equation (4.1) as the flux of the four-vector N^α:

$$N(\Sigma) = \int \sigma_\alpha N^\alpha;$$

because the volume element on $\Sigma = v \times w$, is w and we have noted:

$$\sigma_\alpha = (-g)^{1/2} \epsilon_{\alpha\beta\mu\nu}\, \mathrm{d}x^{\beta\mu\nu}/3!.$$

Likewise, the total momentum contained in Σ is the flux of the tensor $T^{\alpha\beta}$ defined in equation (5.3).

5.2. *Conservation laws*

By means of the equations (4.3)–(4.7), and of the Stokes' theorem, we obtain the conservation law for the numerical flux-vector:

$$N^\alpha{}_{;\alpha} = \int_w C[f]\pi.$$

If the collisions conserve the number of particles then $C[f]$ is given by equation (4.7) and:

$$N^\alpha{}_{;\alpha} = 0. \tag{5.4}$$

One obtains the value of $T^{\alpha\beta}{}_{;\beta}$ by applying the previous result to:

$$G^\alpha = \int_w gp^\alpha \pi/p \tag{5.5}$$

where g is a function of x^α and p^α. As:

$$G^\alpha{}_{;\alpha} = \int_w \left(L(g) + g\frac{\partial(f^\alpha + F^\alpha)}{\partial p^\alpha} \right) \pi \tag{5.6}$$

one obtains $T^{\alpha\beta}{}_{;\beta}$ by writing $g = v_\alpha p^\alpha f$ where v_α is a covariant vector field independent of p^α and such that $v_{\alpha;\beta} = 0$ (see Ehlers 1971):

$$T^{\alpha\beta}{}_{;\beta} = \int_w p^\alpha C[f] + \int_w f(F^\alpha + f^\alpha)\pi.$$

For elastic as well as inelastic collisions ($p^\alpha + q^\alpha = p'^\alpha + q'^\alpha$), the symmetry properties of the collision term (4.7), give:

$$T^{\alpha\beta}{}_{;\beta} = \int_W f(F^\alpha + f^\alpha)\pi. \tag{5.7}$$

Even when there are no external forces ($F^\alpha = 0$), we will not have $T^{\alpha\beta}{}_{;\beta} = 0$, which is normal as the system is open. Note than in equation (5.7) we have supposed that there was no exchange of rest mass with the surroundings during the collision process. This simplifying process is due to the restriction of two-body reactions and to the supposedly instantaneous reaction of the collisions.

Even if $T^{\alpha\beta}$ preserves the usual form of a perfect gas, and even if $F^\alpha = 0$, the presence of f^α will *a priori* entail shearing and vortices.

5.3. Entropy, entropy inequality

The four-vector entropy S^α is defined as usual by:

$$S^\alpha = -k \int_W f(\ln f - 1)p^\alpha \pi/p. \tag{5.8}$$

If one applies equations (5.5) and (5.6) with $g = -kf(\ln f - 1)$, we obtain:

$$S^\alpha{}_{;\alpha} = -k \int_W \ln f\, C[f]\pi + k \int_W f \frac{\partial(f^\alpha + F^\alpha)}{\partial p^\alpha}\pi.$$

A classical calculation shows that the first integral is always greater than or equal to 0, so we obtain for the entropy the inequality:

$$S^\alpha{}_{;\alpha} = k \int_W f \frac{\partial(f^\alpha + F^\alpha)}{\partial p^\alpha}\pi. \tag{5.9}$$

A process will be called reversible if equality occurs, and irreversible if there is strict inequality. Even where there are no external forces, we do not have $S^\alpha{}_{;\alpha} \geqslant 0$, because the dissipative property of the rest mass variation leads to the existence of a supplementary source of entropy.

The reversible condition imposes the same condition on the distribution function as in the usual case, i.e.:

$$\int_W \ln f\, C[f]\pi = 0, \tag{5.10}$$

which indicates that $\ln f$ is a collisional invariant, i.e.:

$$f(x^\alpha, p^\alpha)f(x^\alpha, q^\alpha) = f(x^\alpha, p'^\alpha)f(x^\alpha, q'^\alpha) \tag{5.11}$$

for collisions such that: $p^\alpha + q^\alpha = p'^\alpha + q'^\alpha$.

6. Study of reversible processes

We will study, henceforth, a monocomponent gas in which the particles have variable rest mass and only collide in such a way that:

$$p^\alpha + q^\alpha = p'^\alpha + q'^\alpha. \tag{6.1}$$

As we have already seen, such collisions can be elastic as well as inelastic. If one considers the former alone, one must add to equation (6.1) the following conditions:

$$p = p', \qquad q = q'. \tag{6.2}$$

6.1. *Distribution function*

According to the relationship (5.11) which is valid for reversible processes, the distribution function must be:

$$f(x^\alpha, p^\alpha) = a\, e^{\psi},$$

where a is a scalar function of x^α, and ψ is preserved during the collisions. If the particles are only characterized by their momentum, then the condition (6.1) gives:

$$\psi = -\beta_\alpha p^\alpha,$$

where β_α is a time-like, future-oriented four-vector such that $f \to 0$ if $p \to \infty$. It is interesting to write:

$$\beta_\alpha = \beta U^\alpha,$$

where $\beta \to 0$ and U^α verifies $U^\alpha U_\alpha = 1$. The distribution function is then written:

$$f(x^\alpha, p^\alpha) = C\, e^{-\beta U_\alpha p^\alpha}. \tag{6.3}$$

The five factors C, β, U_α introduced in equation (6.3) can only depend on x^α, and will be determined by the five equations of motion (5.4) and (5.7).

Up to now we have only imposed the condition (6.1) i.e. the variation of rest mass is the result of both the action of f^α and of the inelastic conditions. If we impose as well the condition (6.2) i.e. if the variation of rest mass is only due to f^α, then the five previous factors can generally depend upon p, as long as one always has $f \to 0$ for $p \to \infty$. One sees indeed that this is compatible with the reversibility condition (5.11). Nevertheless, if one wants to keep the usual values for β and U^α (temperature, four-velocity of the fluid), C alone will depend on p. We can then write:

$$f(x^\alpha, p^\alpha) = C(p)\, e^{-\beta U_\alpha p^\alpha}, \tag{6.4}$$

where $C(p)$ is a function of p and of course of x^α. We propose to finish by studying the case where the distribution function is given by equation (6.3).

6.2. *Macroscopic quantities*

The macroscopic quantities N^α, $T^{\alpha\beta}$, S^α defined respectively by equations (5.2), (5.3) and (5.8) can be deduced from A given by equation (5.1). Indeed one can easily show that:

$$N^\alpha = -\partial A / \partial \beta_\alpha \tag{6.5}$$

$$T^{\alpha\gamma} = \partial^2 A / (\partial \beta_\alpha \partial \beta_\gamma), \tag{6.6}$$

and that the entropy can be written in terms of N^α and $T^{\alpha\gamma}$:

$$S^\alpha = -k(\ln C - 1) N^\alpha + k \beta_\gamma T^{\alpha\gamma}. \tag{6.7}$$

To obtain A we can use the convenient coordinate system (p, χ, θ, ϕ) already used by Synge (1957):

$$p^0 = p \cosh \chi$$
$$p^1 = p \sinh \chi \sin \theta \cos \phi$$
$$p^2 = p \sinh \chi \sin \theta \sin \phi$$
$$p^3 = p \sinh \chi \cos \theta$$

where $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$, $\chi \in [0, +\infty[$, $p \in [0, +\infty[$†.

In this coordinate system, the volume element of the four-space W of the momenta is:

$$\pi = p^3 \sinh^2\chi \sin \theta \, dp \, d\chi \, d\theta \, d\phi.$$

The product $\beta_\alpha p^\alpha$ is expressed in the coordinate system $\{x^\alpha\}$ where $U^\alpha = (\vec{0}, 1)$ and one then finds for A:

$$A = (4C/\beta)\bar{K}_{1,1}(\beta),$$

where the properties of the modified Bessel functions $K_n(x)$ and of their integrals $\bar{K}_{m,n}(\beta)$ are developed in the appendix. Taking into account equations (6.5)–(6.7) and these properties, one gets:

$$\begin{aligned} N^\alpha &= nU^\alpha \\ T^{\alpha\beta} &= (\rho_0 c^2 + \bar{p})U^\alpha U^\alpha - \bar{p} g^{\alpha\beta} \\ S^\alpha &= sU^\alpha, \end{aligned} \tag{6.8}$$

where we have noted:

$$\begin{aligned} n &= (4\pi C/\beta)\bar{K}_{2,2}(\beta) \\ \rho_0 c^2 + \bar{p} &= (4\pi C/\beta)\bar{K}_{3,3}(\beta) \\ \bar{p} &= (4\pi C/\beta^2)\bar{K}_{2,2}(\beta) \\ s &= -kn(\ln C - 1) + K\beta\rho_0 c^2 \end{aligned} \tag{6.9}$$

We see immediately that the different macroscopic quantities keep the same form as for the usual perfect fluid. The unitary four-vector U^α represents again the four-velocity of the fluid. It is collinear with N^α and S^α, the factors n and s being respectively the number of material points and the entropy per unit volume of the local rest frame. Furthermore, the decomposition of $T^{\alpha\beta}$ brings out the proper energy $\rho_0 c^2$ of unit volume of the local rest frame and the pressure $\bar{p}$.

As seen in equation (6.9), we have the same equation of state of the perfect fluid:

$$\bar{p} = n/\beta. \tag{6.10}$$

this allows us to interpret β as $(kT)^{-1}$ where T is the temperature measured in the local rest frame. The proper density of entropy can then be written:

$$s = -kn \ln C + k\beta(\rho_0 c^2 + \bar{p}). \tag{6.11}$$

† *If* $p \in [p_1, p_2]$, the following integrals of the Bessel functions must be modified.

If β has finite non-null values, one then has (cf appendix):

$$n = \bar{p}\beta = 6\pi^2 C/\beta^4$$
$$\rho_0 c^2 + \bar{p} = (30\pi^2 C/\beta^5) = (5n/\beta), \tag{6.12}$$

or again:

$$n = \bar{p}\beta = \tfrac{1}{4}\beta\rho_0 c^2, \tag{6.13}$$

which gives:

$$s = kn[5 - \ln(n\beta^4/6\pi^2)]. \tag{6.14}$$

Generally, whatever the value of β, a combination of equations (6.9)–(6.11) gives:

$$s = kn\left(\beta\frac{\bar{K}_{3,3}(\beta)}{\bar{K}_{2,2}(\beta)} - \ln\frac{n\beta}{4\pi\bar{K}_{2,2}(\beta)}\right). \tag{6.15}$$

The equations (6.14) and (6.15) replace the Sakur–Tetrode equation which for a relativistic perfect fluid made up of particles with constant rest mass is (cf Marle 1969):

$$s = kn\left(\beta\bar{p}\frac{K_3(\beta\bar{p})}{K_2(\beta\bar{p})} - \ln\frac{n\beta}{4\pi\bar{p}^2 K_2(\beta\bar{p})}\right).$$

6.3. *Equations of motion*

Since the different macroscopic quantities are known, we can now express their conservation laws, equations (5.4) and (5.7):

$$\dot{n} + nU^\alpha{}_{;\alpha} = 0$$
$$(\rho_0 c^2 + \bar{p})\dot{U}^\alpha = (\hat{F}^\beta + \hat{f}^\beta + \bar{p}_,{}^\beta)\gamma^\alpha{}_\beta \tag{6.16}$$
$$\dot{\rho}_0 c^2 + \rho_0 c^2 U^\alpha{}_{;\alpha} = (\hat{F}^\alpha + \hat{f}^\alpha)U_\alpha - \bar{p}U^\alpha{}_{;\alpha}$$

where $\gamma^\alpha{}_\beta = \delta^\alpha{}_\beta - U^\alpha U_\beta$ is the projection operator on the subspace orthogonal to U^α and where we have noted for every vector $V^\alpha(x, p)$

$$\hat{V}^\alpha = \int_w fV^\alpha\pi.$$

If β is finite and not equal to 0 these equations will give, according to equation (6.12):

$$\dot{n} + nU^\alpha{}_{;\alpha} = 0$$
$$(5n/\beta)\dot{U}^\alpha = [\hat{F}^\lambda + \hat{f}^\lambda + (n/\beta)_,{}^\lambda]\gamma^\alpha{}_\lambda \tag{6.17}$$
$$4\left(\frac{n}{\beta}\right)^{\displaystyle\cdot} + \frac{5n}{\beta}U^\alpha{}_{;\alpha} = (\hat{F}^\alpha + \hat{f}^\alpha)U_\alpha.$$

By combining the first and third of these three relationships, we find:

$$\frac{\mathrm{d}}{\mathrm{d}s}(\ln n\beta^4) = -\frac{\beta}{n}(\hat{F}^\alpha + \hat{f}^\alpha)U_\alpha. \tag{6.18}$$

This equation is the new law of adiabatic processes, which was for $m_0 = cte$ (Synge 1957):

$$n\frac{\beta\bar{p}}{K_2(\beta\bar{p})}\exp\left(-\frac{\beta\bar{p}K_3(\beta\bar{p})}{K_2(\beta\bar{p})}\right) = cte,$$

or for $\beta\bar{p} \gg 1$:

$$n\beta^{3/2} = cte.$$

In the case where the external forces F^α are null and where the exchange of mass is isotropic with the law of equation (2.8), one gets, after a calculation identical to the one giving N^α:

$$\hat{f}^\alpha = -32\pi\lambda C U^\alpha / \beta^5.$$

If one replaces this relationship in (6.18) one finds using the first part of equation (6.12) that:

$$\frac{\mathrm{d}}{\mathrm{d}s}(\ln n\beta^4) = \frac{16\lambda}{3\pi}. \tag{6.19}$$

This gives a simple first integral for the adiabatic processes.

After integration, equation (5.9) gives the variation of the entropy for reversible processes:

$$\dot{s} + sU^\alpha{}_{;\alpha} = k\beta(\hat{F}^\alpha + \hat{f}^\alpha)U_\alpha$$

by combining (6.11), (6.16) and this last equation one checks that:

$$\dot{\rho}_0 c^2 = \frac{\dot{s}}{k\beta} + \frac{\ln C}{\beta}\dot{n}.$$

Let us introduce the rest mass $\bar{m}$ and the proper internal energy ϵ per material point, i.e. let us break down $\rho_0 c^2$ according to:

$$\rho_0 c^2 = n\bar{m}c^2 + n\epsilon.$$

Let us introduce in the same way the entropy η per material point:

$$s = n\eta.$$

One then obtains the law of radiation for internal energy:

$$\dot{\epsilon} = (\dot{\eta}/k\beta) - \bar{p}(\dot{1/n}) = \dot{\bar{m}}c^2.$$

In addition to the first two terms of the right-hand side of this equation whose significance is obvious, the last term represents the contribution of the variation of mass to the internal energy.

7. Conclusion

The modifications of the relativistic kinetic theory which are brought about by the variation of the rest mass of the particles are direct consequences of the dissipative nature of the phenomenon. Among others the volume element of the phase space is no longer preserved, and consequently the Boltzmann equation is slightly modified. The evolution of the system is that of an open system, and kinetic theory has helped us to express this evolution in terms of the action which modifies the rest mass of the particle itself in (5.7). Furthermore, we have been able to clarify the expression of the supplementary entropy source due to the variation of mass, and we have shown that the macroscopic quantities and their evolution laws are similar to those of the usual case.

Appendix. Integrals of the Bessel functions $K_n(x)$

For references connected with this section, see Watson (1966) and Luke (1962). The properties of the modified Bessel functions $K_n(x)$ defined by:

$$K_n(x) = \frac{x^n}{1.3.5\ldots(2n-1)} \int_0^\infty \exp(-x \cosh \chi) \sinh^{2n}\chi \, d\chi \tag{A.1}$$

have already been used in different studies on the relativistic kinetic theory of a system of particles with constant rest mass. It will be noted that if we write $K'_n(x)(dK_n/dx)$, we obtain the following equations:

$$\begin{aligned} &xK'_n(x) - nK_n(x) = -xK_{n+1}(x) \\ &xK'_n(x) + nK_n(x) = -xK_{n-1}(x) \\ &K_{n+1}(x) - K_{n-1}(x) = (2n/x)K_n(x). \end{aligned} \tag{A.2}$$

Here, we need integrals defined by:

$$K_{m,n} = \int_0^\infty x^m K_n(x) \, dx \tag{A.3}$$

or more exactly we have to find integrals of the type:

$$\bar{K}_{m,n}(\beta) = \int_0^\infty t^m K_n(\beta t) \, dt. \tag{A.4}$$

If β is non-null and finite, we can easily see that:

$$\bar{K}_{m,n}(\beta) = \beta^{-(m+1)} K_{m,n}. \tag{A.5}$$

Furthermore, we know that for $m \pm n > -1$,

$$K_{m,n} = 2^{m-1}\Gamma[\tfrac{1}{2}(m+n+1)]\Gamma[\tfrac{1}{2}(m-n+1)] \tag{A.6}$$

where $\Gamma(x)$ is the Eulerian function of the second kind. Equation (A.6) will then give a simple expression for $\bar{K}_{m,n}(\beta)$ when β is non-null and finite.

By using the definition (A.4), one obtains properties similar to (A.2) for $\bar{K}'_{m,n}(\beta) \equiv d(\bar{K}_{m,n}(\beta))/d\beta$

$$\begin{aligned} &\beta\bar{K}'_{m,n}(\beta) - nK_{m,n}(\beta) = -\beta\bar{K}_{m+1,n+1}(\beta) \\ &\beta\bar{K}'_{m,n}(\beta) + n\bar{K}_{m,n}(\beta) = -\beta K_{m+1,n-1}(\beta) \\ &\bar{K}_{m,n+1}(\beta) - \bar{K}_{m,n-1}(\beta) = (2n/\beta)\bar{K}_{m-1,n}(\beta). \end{aligned} \tag{A.7}$$

In this paper, we use only the properties (A.5), (A.6) and (A.7). With the shape chosen for the distribution function, i.e. $f = C\,e^{-\beta_\alpha p^\alpha}$, we will only need values $m = 1, 2, 3$ and $n = 1, 2, 3$. As we have previously mentioned it will be necessary to use other integrals on the lines of the function $C(p)$, if C depends on p in the distribution function.

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